

# Gravitational Lensing of Supernova Neutrinos

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The black hole at the center of the galaxy is a powerful lens for supernova neutrinos. In the very special circumstance of a supernova near the extended line of sight from Earth to the galactic center, lensing could dramatically enhance the neutrino flux at Earth and stretch the neutrino pulse.

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## I. INTRODUCTION

Surely neutrinos—in common with other forms of matter and energy—experience gravitational interactions [1]. Where is the observational evidence to support this assertion? No analogue of the classic Einstein [2]–Eddington [3] demonstration of the deflection of starlight by the Sun is in prospect. No continuous intense point source of neutrinos is known, and the angular resolution of neutrino telescopes—a few degrees achieved at Super-Kamiokande in the few-MeV range and approximately  $\frac{1}{2}^\circ$  projected for km<sup>3</sup>-scale ultrahigh-energy neutrino telescopes—is poorly matched to the anticipated 1.75-arcsecond deflection of neutrinos from a distant source. Accordingly, we must look elsewhere.

Neutrino oscillations arise from phase differences in the propagation of different *inertial*-mass eigenstates. Equivalence-principle-violating models—massless or mass-degenerate neutrinos, with gravity coupling nonuniversally to different flavors—give a poor description of neutrino-oscillation phenomena [4].

The arrival time of the Supernova 1987A neutrino burst, recorded within three hours of the associated optical display after a 166 000-year voyage, argues that neutrinos and photons follow the same trajectories in the gravitational field of our galaxy, to an accuracy better than 0.5% [5, 6]. For a selection of modern halo profiles for our galaxy [7], the Shapiro time delay [8] for photons arriving from SN1987A ranges between  $(9.26 - 11.5) \times 10^6$  s = 0.29 – 0.36 y. [Observe that, for small neutrino masses, neutrinos lag light by only  $\Delta t \approx \frac{1}{2}(m_\nu/p_\nu)^2 t \approx 2.5 \mu\text{s} \cdot (m_\nu/1 \text{ eV})^2 / (p_\nu/1 \text{ GeV})^2$  over the SN1987A–Earth trajectory.]

Even if they experience normal gravitational interactions, neutrinos do not cluster gravitationally on small scales, because of their large velocities. Free-streaming neutrinos inhibit the growth of density fluctuations, and so leave an imprint on the large-scale-structure matter-power spectrum that is directly related to the neutrino

energy density [9]. Weak-lensing surveys offer the most promising path to precise measurements of the matter-power spectrum and might, in the future, be sensitive to a nonzero (inertial) neutrino mass [10, 11, 12].

The SN1987A argument, though telling, is indirect. Can we imagine more direct manifestations of gravity’s influence on neutrinos?

If it could be carried out, a neutrino analogue [13] of the Pound–Rebka experiment [14], applying the Mössbauer effect to recoilless resonant capture of antineutrinos [15, 16], would demonstrate the blue shift of neutrinos falling in a gravitational field.

In this Article, we explore the possibility that gravitational lensing of neutrinos could be observed in special circumstances. Neutrinos emitted by a supernova on the other side of our galaxy would be lensed by the black hole at the galactic center. In the exceedingly rare circumstance of source–lens–observer syzygy, the flux of neutrinos arriving at Earth would be amplified by many orders of magnitude. For the somewhat less improbable case of near-perfect opposition, lensed neutrinos would reach Earth by paths of differing lengths, resulting in an observable time dispersion of the supernova neutrino burst.

After computing the amplification and time dispersion for various configurations, we estimate the rate at which appropriately providential circumstances might occur. We assess the diffusing influence of the dark-matter halo in the galaxy. We briefly consider signatures of possible megamagnifications throughout the history of the solar system and in present-day neutrino observatories, and we remark on lensing by nearby stars.

## II. LENSING PRIMER

General Relativity predicts that a light ray (or neutrino) that passes by a spherical body of mass  $M$  is deflected by an angle [2, 17, 18]

$$\alpha = \frac{4GM}{c^2\xi} = \frac{2\mathcal{R}}{\xi} \quad (1)$$

for an impact parameter  $\xi$  much larger than the Schwarzschild radius  $\mathcal{R} = 2GM/c^2$ , where  $G =$

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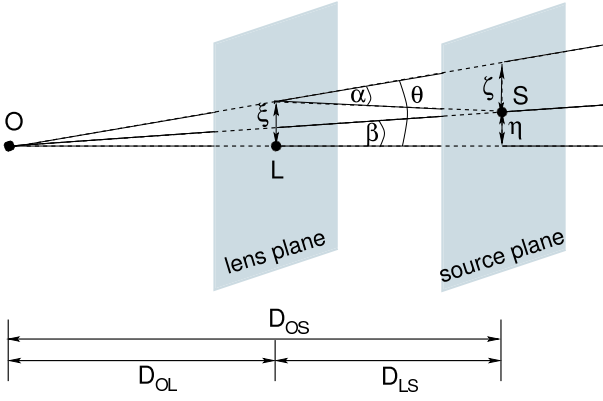


FIG. 1: Lensing geometry

$6.6742 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$  is Newton's constant and  $c = 299\,792\,458 \text{ m s}^{-1}$  is the speed of light. [Values not otherwise attributed are taken from the *Review of Particle Physics* [19].] For the special case of a ray that skims our Sun, with impact parameter  $\xi$  equal to the solar radius  $R_\odot = 6.961 \times 10^5 \text{ km}$  and  $\mathcal{R}_\odot = 2.953 \text{ km}$ , the angle of gravitational deflection is  $\alpha = 8.48 \mu\text{rad}$ . As we have already noted, such a subtle deviation is unobservably small for neutrinos. The black hole at the center of the Milky Way galaxy, with mass  $M_\bullet = (3.61 \pm 0.32) \times 10^6 M_\odot \approx 4 \times 10^{63} \text{ GeV}$  [20] and Schwarzschild radius  $\mathcal{R}_\bullet = 1.07 \times 10^7 \text{ km} \approx 3.46 \times 10^{-7} \text{ pc}$ , is a better candidate for a neutrino lens.

In the simplest lensing set-up, shown in Figure 1, a compact lens of mass  $M$  lies close to the line of sight to a source, at a distance  $D_{OL}$  from the observer  $O$ . The angle  $\beta$  describes the position of the source with respect to the lens direction. The angle  $\theta$  describes the position of the apparent source image with respect to the same axis.

If all the angles are very small, it is appropriate to define angular-diameter distances  $\eta = \beta D_{OS}$ ,  $\xi = \theta D_{OL}$ , and  $\zeta = \alpha D_{LS}$ . In this approximation, we can infer from Figure 1 the lens equation,

$$\beta = \theta - \frac{\theta_E^2}{\theta}, \quad (2)$$

where the Einstein angle  $\theta_E$  is

$$\theta_E = \sqrt{\frac{2\mathcal{R}D_{LS}}{D_{OS}D_{OL}}} = \sqrt{\frac{2\mathcal{R}}{D_{OL}} \frac{x}{1+x}}, \quad (3)$$

with  $x \equiv D_{LS}/D_{OL}$ . [For the interesting case of the black hole at the center of the galaxy, 8.0 kpc distant from our location [21], the Einstein angle would be  $\theta_E = 3.1 \mu\text{rad}$  for a source 1 kpc beyond the galactic center,  $\theta_E = 6.6 \mu\text{rad}$  for a source opposite our location, and  $\theta_E = 7.6 \mu\text{rad}$  for a source at the far edge of the galaxy ( $x = 2$ ).] In the plane OLS, the extremal angles of deflection are given by

$$\theta_\pm = \frac{\beta}{2} \pm \theta_E \sqrt{1 + \beta^2/4\theta_E^2}. \quad (4)$$

Sufficiently strong lensing produces multiple images of the source. If the source, lens, and observer lie on a line ( $\beta = 0$ ), the multiple images describe a perfect circle, or Einstein ring, with opening angle  $2\theta_E$ . More generally, finite-size lenses give rise to a variety of image patterns, depending on the particularities of the mass distribution within the lens. Distinct multiple images, arcs, or Einstein rings have been observed for many light sources [22]. The separation between images—no more than a few arcseconds—cannot be resolved in neutrinos.

In many cases the lens is not strong enough to produce multiple images, arcs, or rings, but does create a distorted image of the source. Usually the precise sizes and shapes of the sources are not known, but it is possible to characterize *average* properties. The “weak lensing” method [23]—comparing statistics of sources and images—has been used to weigh nearby clusters by using distributions of faraway galaxies as sources. Weak lensing is also a powerful tool in cosmology: statistics of the large scale structure observed in distant galaxies or the cosmic microwave background make it possible to infer the total (luminous plus dark) mass between the observer and the source. Large surveys—not in prospect for neutrinos—are essential to the statistical reliability of inferences.

In the “microlensing” case [24], multiple images are overlaid too closely to be resolved and image distortion is too subtle to observe, but light reaching the observer along multiple trajectories increases the apparent brightness of the source. Without knowing the intrinsic brightness of the source, it is generally not possible to determine the amplification, or “magnification,” as it is usually called. One can, however, observe the time variation of the brightness of a source (such as a nearby star), as a heavy object passes between source and observer. Microlensing in this form is the basis of searches for massive compact halo objects (MACHOs) [25].

The magnification is given by the ratio of solid angles in the presence and absence of the lens. At the extreme angles, we have

$$\mu_\pm = \frac{d\Omega}{d\Omega_0} = \left| \frac{\theta_\pm}{\beta} \frac{d\theta_\pm}{d\beta} \right|, \quad (5)$$

whereupon (cf. (4))

$$\mu = \mu_+ + \mu_- = \frac{1 + \frac{1}{2}\bar{\beta}^2}{\bar{\beta}\sqrt{1 + \frac{1}{4}\bar{\beta}^2}} \quad (6)$$

where  $\bar{\beta} \equiv \beta/\theta_E$  is the reduced misalignment angle. The magnification is maximized in the limit of perfect alignment, as  $(\beta, \bar{\beta}) \rightarrow 0$ , for which

$$\mu \rightarrow \mu^{\max} = 1/\bar{\beta}. \quad (7)$$

For a finite source of radius  $R_\star$ , the effective limit is  $\beta \rightarrow \beta_\star = R_\star/D_{OS}$ . In that limit, the magnification can be prodigious: for a source of radius  $R_\star = 10 \text{ km}$  on the

other side of the galaxy lensed by the black hole at the galactic center, we find

$$\mu^{\max} = 2.3 \times 10^{11} \sqrt{x(1+x)}. \quad (8)$$

In comparison to the  $\beta$  dependence of the amplification, the dependence on the source location is of secondary importance: as  $x$  varies between 0.01 and 2,  $\mu^{\max}$  varies only by a factor of 25.

The calculation we have just made applies to a fictitious galaxy that is empty except for the source, the black hole at the galactic center, and the observer. In the real Milky Way, matter obscures visible light from the other side of the galaxy, but the magnification of gamma-ray, radio, or neutrino sources might be observable. [We shall verify in §IV that diffuse matter throughout the galaxy contributes negligibly to lensing.]

Superposed lensed images arise from neutrinos that traverse different paths, and so signals from the source reach the observer at different times, as Krauss and Small have remarked [26] for the microlensing of light. [The time delay between the arrival of neutrinos traveling on different trajectories has been invoked in [27] to explain a putative bimodal time distribution of SN1987A neutrinos.]

It is convenient to calculate the transit time along each geodesic by adding the interval from source to lens to the interval from lens to observer. The propagation time has two components [28]: one corresponds to the time required for straight-line propagation to and from the lens, and the second is a general-relativistic time delay [8] proportional to the Schwarzschild radius of the lens. For our case of a black-hole lens, integration along the curved geodesic yields the exact result,

$$\begin{aligned} ct = & \sqrt{r_O^2 - \xi^2} + \sqrt{r_S^2 - \xi^2} \\ & + \frac{\mathcal{R}}{2} \left[ \sqrt{\frac{r_O - \xi}{r_O + \xi}} + 2 \ln \left( \frac{r_O + \sqrt{r_O^2 - \xi^2}}{\xi} \right) \right. \\ & \left. + \sqrt{\frac{r_S - \xi}{r_S + \xi}} + 2 \ln \left( \frac{r_S + \sqrt{r_S^2 - \xi^2}}{\xi} \right) \right] \quad (9) \end{aligned}$$

where  $r_O = D_{OL}$  is the distance from the lens to the observer,  $r_S = \sqrt{D_{LS}^2 + \eta^2} = D_{OL} \sqrt{x^2 + \beta^2(1+x)^2}$  is the distance from the lens to the source,  $\xi$  is the distance of closest approach to the lens, and  $\mathcal{R}$  is the Schwarzschild radius of the lens. For a source 8 kpc beyond the galactic center, and with misalignment angle  $\beta = \theta_E$ , the relativistic time delay induced by the black hole is  $O(10^3)$  s.

The longest geodesic corresponds to  $\xi_- = D_{OL}|\theta_-|$  and the shortest one to  $\xi_+ = D_{OL}\theta_+$ , with  $\theta_{\pm}$  given in Equation (4). Light signals emitted at the same time can travel by different paths, so they reach an observer separated in coordinate time by as much as

$$\Delta t \equiv (t_- - t_+) \quad (10)$$

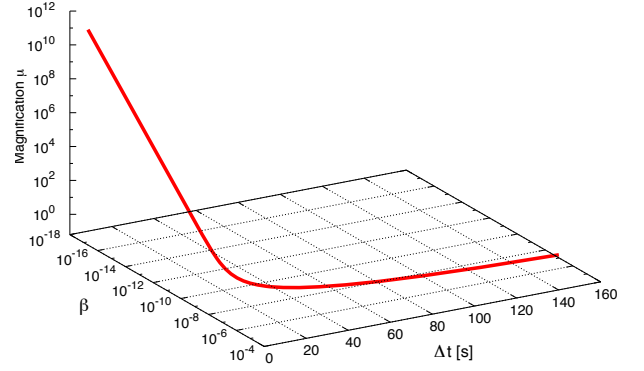


FIG. 2: Magnification  $\mu$  and dispersion in time  $\Delta t$  as a function of misalignment angle  $\beta$ , for the case of a 10-km-radius source opposite our location in the galaxy.

$$= \frac{\mathcal{R}}{c} \left[ 2\bar{\beta} \sqrt{1 + \frac{1}{4}\bar{\beta}^2} + \ln \left| \frac{1 + \frac{1}{2}\bar{\beta}^2 + \bar{\beta} \sqrt{1 + \frac{1}{4}\bar{\beta}^2}}{1 + \frac{1}{2}\bar{\beta}^2 - \bar{\beta} \sqrt{1 + \frac{1}{4}\bar{\beta}^2}} \right| \right].$$

[An observer registers a proper interval  $\Delta\tau = \sqrt{g_{00}}\Delta t$ ; for an earthbound observer,  $g_{00}$  differs negligibly from unity.] In the limit of very small misalignments ( $\beta \rightarrow 0$ ), the time dispersion is proportional to  $\bar{\beta}$ , viz.

$$\Delta t \rightarrow \frac{\mathcal{R}}{c} \left[ 2\bar{\beta} + \ln \left| \frac{1 + \bar{\beta}}{1 - \bar{\beta}} \right| \right] \approx \frac{4\mathcal{R}\bar{\beta}}{c} = \frac{2\beta}{c} \sqrt{2\mathcal{R}D_{OL} \frac{1+x}{x}}, \quad (11)$$

so that the magnification and time dispersion are reciprocally related through

$$\mu \Delta t = 4\mathcal{R}/c \approx 142 \text{ s}. \quad (12)$$

In the small- $\bar{\beta}$  limit, both the magnification and the time spread scale with the square root of the lens mass. For the configuration considered below Eqn. (7), the time spread would be  $\Delta t \approx 4.4 \times 10^{-10}$  s. At the other extreme, for  $\bar{\beta} = 1$ ,  $\Delta t \approx 148$  s and  $\mu = 3/\sqrt{5} = 1.34$ .

Figure 2 shows the magnification and time dispersion as a function of the misalignment angle for  $\beta_* \leq \beta \leq \theta_E$ . For  $\beta \ll \theta_E$ , we observe the  $\mu \propto \beta^{-1}$  behavior of Eqn. (7). As  $\beta$  increases toward  $\theta_E$ , the magnification diminishes and the dispersion in time increases to many seconds.

### III. GALACTIC SUPERNOVA LENSED BY THE BLACK HOLE AT THE GALACTIC CENTER

#### A. General Characteristics

The lensing phenomena we have described for light signals (in a hypothetical nearly empty galaxy) apply essentially unchanged for neutrinos, with utterly negligible corrections for the neutrino velocity, which is slower than the speed of light by  $\frac{1}{2}(m_\nu/p_\nu)^2 \cdot c$ . The Milky Way galaxy is highly transparent to low-energy neutrinos. The  $\nu N$  interaction length at  $E_\nu = 10$  MeV, for

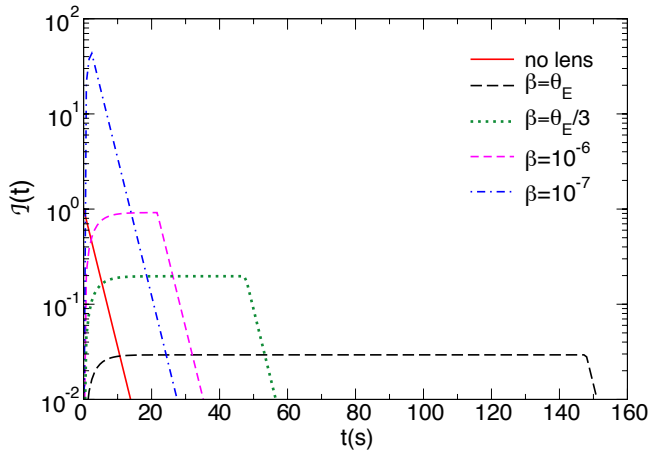


FIG. 3: Normalized neutrino intensity as a function of time for lensed supernovae with  $\beta = \theta_E$  (black dashed curve),  $\beta = \theta_E/3$  (green dotted),  $\beta = 10^{-6}$  (magenta short dashes),  $\beta = 10^{-7}$  (blue dot-dashed). The no-lensing case is shown as the red solid curve.

example, exceeds  $10^{17}$  cmwe, whereas an average diameter through the galactic disc integrates a column density considerably less than 1 cmwe [29]. Subtle effects due to neutrino lensing by stars (including the Sun) and galactic halos have been identified in Ref. [30]. Here we analyze a more dramatic illustration of gravitational lensing: A supernova at superior conjunction to the black hole at the galactic center would be an ideal source for the study of gravitational lensing of neutrinos.

The gravitational binding energy  $E_B = 3 \times 10^{53}$  erg of a core-collapse supernova is roughly equipartitioned among the six neutrino and antineutrino species. A useful description of the supernova luminosity in neutrinos consists of a nearly instantaneous rise followed by an exponential decay [31, 32] that can be represented by

$$L_0(t) = \frac{E_B}{6\tau} \exp^{-t/\tau} \equiv L_0 \exp^{-t/\tau}. \quad (13)$$

The decay time  $\tau = 3$  s implies an effective pulse length of  $\approx 10$  s, consistent with SN 1987A observations.

A time spread of neutrinos arriving at Earth from a lensed supernova that considerably exceeds the canonical 10-s pulse length would be a signature of lensing. We plot in Figure 3 some examples of the time profiles to be expected for a supernova at distance  $D_{OS} = 16$  kpc, for some representative values of the misalignment angle  $\beta$ .

In the absence of lensing, the intensity of neutrinos arriving at Earth as a function of time is an exponentially decaying pulse proportional to  $L_0(t)$ . When the supernova neutrinos are lensed, neutrinos may arrive by different paths, with different travel times. For very small misalignment angles, the difference in path lengths is very small, and so the resulting intensity profile is indistinguishable from  $\mu \times$  the unlensed profile. An example is given in the curve corresponding to  $\beta = 10^{-7}$  in

Figure 3, which is very nearly given by  $\mu = 65.7$  times the unlensed intensity profile. The normalization of the intensity curves  $\mathcal{I}(t)$  plotted in Figure 3 is such that  $\int_0^\infty dt \mathcal{I}(t) = \mu\tau$ .

As the misalignment angle  $\beta$  increases toward  $\theta_E$ , new and longer paths contribute, so the intensity is nearly constant over an increasingly long time interval. Thereafter, no new paths come into play and the intensity decays according to the exponential in  $L_0(t)$ , but displaced in time. The onset of this distortion of the pulse is shown by the  $\beta = 10^{-6}$  curve in Figure 3; the fully developed behavior is exhibited for the case  $\beta = \theta_E$ . For values of  $\beta \approx \theta_E/3$ , both the magnification and the time delay are significant.

## B. Likelihood of lensing events

The near-perfect alignment of source, lens, and observer is a very special circumstance. How frequently might a dramatic lensing event occur?

Let us first assume that supernovae are distributed according to the mass density in the galactic disc [33],

$$\sigma(r) = \sigma_0 e^{-r/r_0}, \text{ for } r \leq r_G, \quad (14)$$

where  $r$  is the distance from the galactic center,  $r_G = 15$  kpc is the galactic radius, and the parameter  $r_0 = 3.5$  kpc. [It is an excellent approximation for our purposes to idealize the disc as infinitely thin, so that  $\sigma(r)$  measures the luminous mass density per unit area.] A useful measure of the fraction of supernovae that lie along a swath at radius  $r = D_{LS} = x D_{OL}$  is then

$$f(r) = \frac{2\pi\sigma(r)r\Delta r}{2\pi \int_0^{r_G} dr r \sigma(r)}, \quad (15)$$

where  $\Delta r = 2R_\star \approx 20$  km is a typical supernova diameter. In terms of the dimensionless quantities  $x$  and  $x_0 = r_0/D_{OL} = 3.5/8$ , we find

$$f(x) = 4.6 \times 10^{-16} x e^{-x/x_0}. \quad (16)$$

Only a tiny fraction of the supernovae that occur at a given radius are aligned closely enough with observer and lens—with a detector on Earth and the black hole at the center of the galaxy—to be observably lensed. A reasonable measure is the ratio  $\theta_E/\pi$ , which leads to the probability that a galactic supernova at reduced radius  $x$  be lensed toward Earth of

$$P(x) = f(x) \cdot \frac{\theta_E}{\pi} \approx 1.35 \times 10^{-21} x e^{-x/x_0} \sqrt{\frac{x}{1+x}}. \quad (17)$$

Integrating over all radii, we find that the fraction of supernovae lensed toward Earth is approximately  $1.8 \times 10^{-6}$ .

It has been argued that while the distribution of Type IA supernovae closely tracks the luminous matter, the



distribution of neutron stars or pulsars is a better tracker of the core collapse supernovae that concern us here [34]. Under either assumption, the density has the form

$$\sigma(r) = \sigma_0 r^p e^{-r/r_0}, \text{ for } r \leq r_G, \quad (18)$$

where ( $p = 4$ ,  $r_0 = 1.25$  kpc) for neutron stars and ( $p = 2.35$ ,  $r_0 = 1.528$  kpc) for pulsars. The radius at which the density of supernovae is greatest is approximately 5 kpc for the neutron-star model and 3.5 kpc for the pulsar model.

If core-collapse supernovae track the neutron-star distribution, we compute the fraction of supernovae that lie along a swath at radius  $r = D_{\text{LS}} = x D_{\text{OL}}$  as before:

$$f(x) = 4.7 \times 10^{-14} x^5 e^{-x/x_0}, \quad (19)$$

where now  $x_0 = r_0/D_{\text{OL}} = 1.25/8$ . Now the probability that a galactic supernova at reduced radius  $x$  be lensed toward Earth is

$$P(x) \approx 1.4 \times 10^{-19} x^5 e^{-x/x_0} \sqrt{\frac{x}{1+x}}. \quad (20)$$

The fraction of galactic supernovae lensed toward Earth is  $2 \times 10^{-6}$ , quite comparable to the fraction we found if the supernovae are assumed to follow the visible matter distribution. [The fraction for the pulsar distribution is  $1.9 \times 10^{-6}$ .] We show in Figure 4 the probability that a lensed supernova be amplified by a factor  $\mu$  or greater, if the supernovae sites track the distribution of neutron stars. Our other two hypotheses yield very similar results.

The current rate of galactic supernovae that produce neutrino bursts is estimated [35] at one in  $47 \pm 12$  y. If the rate of supernova neutrino bursts throughout the galaxy is constant over time at the current rate of approximately two per century, we conclude that on the order of 180 lensed supernovae have occurred throughout the 4.5-Gy history of the solar system. The time between lensing events with amplification factor  $\mu = 10$  is approximately 250 million years, and an event with  $\mu = 100$  has occurred perhaps once in the history of the solar system. The frequency of events with magnifications  $\mu \geq 10$  is comparable to the rate of nearby supernova explosions [36, 37, 38, 39, 40]. These are conservative estimates, given that the supernova rates are not constant over time and were apparently higher when the galaxy was young [41].

It is also interesting to ask how many supernovae exhibit a noticeably increased pulse-length as a consequence of gravitational lensing. The fraction of supernovae for which the time dispersion lies between 50 s, for which  $\mu = 2.99$ , and the value (148 s, with  $\mu = 1.34$ ) corresponding to  $\beta = \theta_E$  is  $1.3 \times 10^{-6}$ . Over the history of the solar system, supernova neutrino pulses longer than 50 s have arrived at Earth approximately 117 times—an average of one every 38.5 million years. Lowering the requirement to a time dispersion of 20 s, for which  $\mu = 7.16$ , increases the fraction to  $1.72 \times 10^{-6}$ . This rate

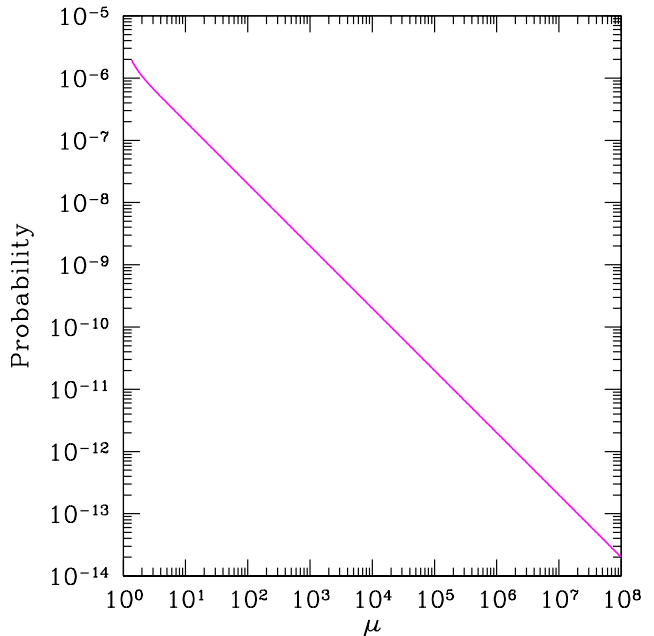


FIG. 4: Probability that lensing of a core-collapse supernova by the black hole at the center of the galaxy amplifies the neutrino flux at Earth by a factor  $\mu$ , assuming that the distribution of supernovae tracks the population of neutron stars in the galactic disc.

corresponds to 155 stretched supernova neutrino pulses over the age of the solar system, roughly one every 29 million years.

### C. Signatures in Neutrino Observatories

A few dozen  $\bar{\nu}_e$  from SN1987A were recorded in a number of detectors (IMB [42], Kamiokande II [43], Baksan [44], and Mont Blanc [45]) via the charged-current process  $\bar{\nu}_e + p \rightarrow e^+ + n$ . These observations set the stage for the detection of neutrinos from future supernovae. [It is less certain [46] that  $\nu_e$ -initiated events associated with SN1987A have been established. The detection of a single in-time  $\nu_e e$  event would test the equivalence principle for  $\nu_e$  vs.  $\bar{\nu}_e$  to a few parts in a million [47, 48].]

The signatures of a core-collapse supernova lensed by the black hole at the galactic center are a significant amplification of the neutrino flux at the detector and a dispersion in time of the neutrino burst. The larger each of these effects—which are correlated if lensing occurs—the smaller is the likelihood that the supernova is simply an outlier. Pointing information derived from neutrino interactions is crude, but to implicate lensing it would suffice to identify the supernova direction along a line through the galactic center.

The yield of neutrinos emitted by a core-collapse supernova can be anticipated within a factor of two to

three [49]. Under that assumption, a measurement of the distance to the supernova provides a good estimate of the unlensed neutrino flux. For supernovae opposite our location in the galaxy, it is likely that visible light would be so attenuated traversing the matter in the galactic disc that the precursor star could not be identified. The Chandra mission, with its angular resolution of  $2.5 \mu\text{rad}$ , has catalogued many x-ray point sources toward the galactic center [50].

By measuring the neutrino flux alone, one could infer an apparent distance to an unlensed source. With a flux greatly magnified by lensing, the apparent distance might be implausibly small, placing the supernova on this side of the galactic center, where the optical signal would have been visible. Lensing would then be strongly implicated, even without a precise determination of the magnification.

We summarize in Table I some characteristics for supernova neutrino detection of the available techniques, which entail several elements: sensitivity to multiple neutrino flavors (through neutral-current measurements), timing, energy resolution, and pointing back to the source (through  $\nu e$  elastic scattering [32]). The most interesting detector capabilities for supernova lensing applications are the time resolution and the capability to point to the source. All of the techniques listed in Table I allow for an adequate measurement of arrival time. We will focus on water Cherenkov detectors (such as Super-Kamiokande [51]) and long-string ice/water Cherenkov detectors (such as AMANDA [52]), because they are operating and the next-generation version of AMANDA (IceCube [53]) is under construction.

It is convenient to scale the total number of neutrino events in a detector of given effective mass as [54]

$$N = N_0 \left( \frac{E_B}{3 \times 10^{53} \text{ erg}} \right) \left( \frac{10 \text{ kpc}}{D_{\text{OS}}} \right)^2, \quad (21)$$

where as usual  $E_B$  is the gravitational binding energy of the collapsing star and  $D_{\text{OS}}$  is the distance between observer and source. Assuming sensitivity to all reactions, the reference rate is  $N_0 \approx O(10^4)$  for the Super-Kamiokande detector with 32 kton of  $\text{H}_2\text{O}$  and a 5-MeV threshold. [For the SNO detector [55], which is expected to stop running at the end of 2006,  $N_0 \approx O(10^3)$ .] Therefore, a supernova with a typical 10-s time profile located at the symmetrical configuration ( $D_{\text{OS}} = 16 \text{ kpc}$ ) would generate approximately 3900 total events in a Super-Kamiokande-like detector—in the absence of lensing effects.

If supernova neutrinos are lensed by the black hole at the center of our galaxy, the number of events in a Super-Kamiokande class detector could be ten times greater, with a time dispersion of 15 s, or five times greater, with a time dispersion of 30 s, depending on the value of the misalignment angle  $\beta$ , see Figure 2. The neutrino time profile would be greatly distorted from what would be reconstructed in the absence of lensing: it would be delayed, highly amplified, and broadened, as can be seen

from Figure 3. It is worth noting that Super-Kamiokande can measure up to 30 000 events within the first second of a burst, with no dead time [51].

Among several proposals for future water Cherenkov detectors, UNO (the Ultra underground Nucleon decay and neutrino Observatory) [56], with a fiducial volume twenty times Super-Kamiokande's, would record  $10^5$  neutrino interactions from a supernova located 10 kpc away, which means 39 000 events if  $D_{\text{OS}} = 16 \text{ kpc}$ . In the presence of lensing effects, the number of events recorded in UNO could reach an order of magnitude larger,  $4 \times 10^5$  events with a time dispersion of 15 s, or five times larger,  $2 \times 10^5$  events dispersed over 30 s.

Although the energy of supernova neutrinos lies far below the threshold for track reconstruction in long-string Cherenkov detectors, a supernova neutrino burst would give rise to a coincident rate increase above ambient noise in all photomultipliers in a large array. The shortcoming of this technique is that it gives no information about the direction to the supernova.

IceCube, the successor to AMANDA, is under construction at the South Pole. A supernova 10 kpc away would generate  $1.5 \times 10^6$  excess photoelectrons over 10 s in IceCube's 4800 optical modules, to be compared with a background noise of  $1.44 \times 10^7$  photoelectrons, for a favorable signal-to-noise ratio  $S/\sqrt{N} \approx 400$  [57]. If we rescale to our ideal situation  $D_{\text{OS}} = 16 \text{ kpc}$ , the number of excess photoelectrons would be  $6 \times 10^5$ , for which  $S/\sqrt{N} \approx 160$ . In the presence of lensing, both  $S$  and  $S/\sqrt{N}$  could be considerably enhanced. The standard supernova searches in AMANDA/IceCube bin data on 500 ms–10 s scales because of the typically low signal-to-noise ratio. For a lensed supernova with enhanced signal-to-noise, the AMANDA supernova detection system, which records data over intervals as short as 10 ms, could reconstruct a very precise time spectrum.

## D. Signatures in the Historical Record

Because the probability of witnessing a lensed supernova is so tiny, it is worthwhile to ask whether it might be possible to recognize the effects of a past event, and so greatly increase the integration time for observations. Two possibilities come to mind: isotopic anomalies and doomsday events. Neither seems promising as an unambiguous marker for a supernova lensed toward Earth.

Supernova neutrinos may induce inverse beta decay in nuclei within the Earth. Indeed, the idea of radiogeochemical studies [66] such as the molybdenum-technetium experiment [67, 68] is to infer the supernova rate in the galaxy by integrating the neutrino flux over several millions of years. It exploits a reaction on a naturally occurring ore target that leads to an isotope with a half-life far shorter than the age of Earth,

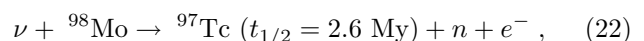


TABLE I: Neutrino observatory techniques and their capabilities for the detection of supernova neutrinos.

Detector type: Examples	Energy resolution	Pointing	Flavor tag
Water Cherenkov: Super-Kamiokande [51], UNO [56]	Yes	Yes	$\bar{\nu}_e$
Long-string Cherenkov: AMANDA [52], IceCube [53], Baikal [58], NESTOR [59], ANTARES [60]	No	No	$\bar{\nu}_e$
Scintillator: KamLAND [61], LVD [62]	Yes	No	$\bar{\nu}_e$
Heavy water: SNO [55]	Yes	Yes	$\bar{\nu}_e$ (CC)
	No	No	all (NC)
High- $Z$ / neutron: OMNIS [63], LAND [64]	Yes	No	all
Liquid Argon: ICANOE [65]	Yes	Yes	$\nu_e, \bar{\nu}_e$

for which the 7.28-MeV threshold excludes most solar neutrinos.

Haxton & Johnson [66] have found that the average flux of neutrinos from galactic supernovae is reproduced by placing all supernovae at 4.6 kpc from Earth. The neutrino flux from a lensed supernova at distance  $8 \text{ kpc} \cdot (1+x)$  is thus

$$R = \frac{\mu}{(1+x)^2} \left( \frac{4.6 \text{ kpc}}{8 \text{ kpc}} \right)^2 \quad (23)$$

times that from a prototypical supernova. The Mo-Tc technique integrates neutrino fluxes over several million years. Over 1 My, at the current rate of 2 galactic supernovae per century, the ratio of neutrinos from a single lensed supernova to the unlensed galactic background is

$$\bar{R} = \frac{R}{2 \times 10^4} = 1.65 \times 10^{-5} \frac{\mu}{(1+x)^2}. \quad (24)$$

Now, for a single, perfectly aligned event, we would find using Eqn. (8) a megayear ratio

$$\bar{R} = 3.8 \times 10^6 \sqrt{\frac{x}{(1+x)^3}} \approx 1.34 \times 10^6, \text{ for } x = 1. \quad (25)$$

A supernova opposite our location in the galaxy would stand out dramatically from the unlensed background.

On the other hand, we have seen that highly magnified supernovae are exceedingly rare, with only a single  $\mu \gtrsim 100$  event having occurred over the lifetime of the solar system. A magnification  $\mu \gtrsim 2 \times 10^6$  would cause a  $3\text{-}\sigma$  excess in the megayear integrated neutrino flux. Such an event is stupendously improbable, occurring only for one galactic supernova in  $10^{12}$ .

Even under such propitious circumstances, the factor-of-three uncertainty in the rate of galactic supernovae, the existence of backgrounds from  $\nu_{\text{solar}} \text{ } ^{97}\text{Mo} \rightarrow \text{}^{97}\text{Tc} + e^-$ , and the possibility that nearby star-forming regions might constitute an uncontrolled background [69], would make it challenging to attribute a larger-than-nominal  $^{97}\text{Tc}$  abundance to a lensing event. A possible discriminant might be found in other isotopic abundances [38, 39] that might carry the imprint of cosmic rays associated with a nearby supernova explosion.

It is natural to wonder whether a nearby supernova might have triggered an extinction episode in the paleontological record. Ellis & Schramm [37] concluded that the  $\gamma$  fluxes and charged cosmic rays from a supernova explosion at 10 pc from Earth—a once in a few hundred million years occurrence—would produce a hole in the ozone layer that would admit lethal solar radiation, dramatically altering photosynthesis cycles. The ionizing radiation that would cause such a cataclysm would not reach the solar system in significant amounts if the supernova were located instead at the other side of the galaxy.

Whether supernova neutrinos would provoke appreciable damage to living organisms is a matter of debate. At issue is whether nuclei recoiling from neutrino collisions would cause irreparable damage to genetic material [40]. Our only comment is that if neutrinos from a nearby supernova did have lethal effects, the neutrino flux from a perfectly aligned supernova opposite our location would be more damaging still. The huge magnification factor of  $\mu \approx 3 \times 10^{11}$  more than compensates the  $r^{-2}$  suppression, making the neutrino flux up to five orders of magnitude greater than in the case of a supernova 10 pc from Earth.

#### IV. OTHER LENSES AND SOURCES

Under the rare circumstances of a supernova opposite our location in the galaxy, lensing of neutrinos by the black hole at the galactic center could produce a prodigious magnification of the neutrino flux at Earth. It is worth considering whether other microlensing effects—induced, for example, by multiple gravitational scattering on the dark matter in the galaxy—might diffuse the supernova neutrinos and so diminish the magnification. These effects are negligible for the case at hand.

The dark matter that could disrupt our conclusions about black-hole lensing is essentially contained within the extremal neutrino geodesics that we consider. To obtain a slightly generous estimate, we compute the dark matter contained within cones of opening angle  $\theta_E$ , directed from source and observer toward the galactic center. For the halo profiles given in Refs. [7, 70, 71, 72], dark matter contained in the “geodesic cones” is less than

one percent of the black-hole mass, an amount that may safely be neglected.

Stars are effective lenses for light, which is observed with excellent angular resolution, but are unlikely to produce observable effects for neutrinos [30, 73], even when finite-lens effects are taken into account.

For the lensing of a distant source by a nearby star,  $D_{OL}$  is relatively small, while  $x = D_{LS}/D_{OL}$  is very large. The magnification for small misalignment angle  $\beta$  is

$$\mu \approx \frac{\theta_E}{\beta} \simeq \frac{\sqrt{2\mathcal{R}D_{OL}}}{R_*}x, \quad (26)$$

where  $\mathcal{R}$  is the Schwarzschild radius of the star and  $R_*$  is the radius of the source. For the case of a core-collapse supernova lensed by  $\alpha$ Cen, approximately one solar mass 1.34 pc distant from Earth,  $\mu \approx 10^6x$ . For a supernova at a distance of 1 Mpc the magnification would be again enormous,  $\mu \approx 10^{12}$ . An unlensed supernova at this distance for which one detected neutrino event would be expected [74] would result in some  $10^{12}$  events if perfectly aligned with  $\alpha$ Cen and Earth. It is thus conceivable that lensing could allow for the detection of significant numbers of neutrinos from more distant supernovae. However, the Einstein angle for  $\alpha$ Cen is nearly twenty times smaller than that for the black-hole scenario, so the probability of perfect alignment is proportionately reduced. No extreme magnification events have been observed for light.

Many other potential neutrino sources, including extremely distant supernovae, gamma ray bursters, and active galactic nuclei, could in principle be greatly magnified because  $x$  is so large for a distant source and nearby lens. The number of such neutrinos reaching Earth from extragalactic sources scales as

$$\frac{\mu}{D_{OS}^2} = \frac{\mu}{D_{OL}^2(1+x)^2} \propto \frac{\mu}{x^2}, \quad (27)$$

which implies low counting rates in the absence of lensing for sources at many megaparsecs. As can be seen from Eq. (26), the magnification compensates one power of  $x$ .

Only for supermassive lenses might the magnification be large enough to overcome the small anticipated flux from a very distant source. We know of no nearby candidates as effective as the black hole at the galactic center. Magnification by distant lenses (for which  $D_{OL}$  is very large) is likely to be modest, because  $x$  is small in that case.

To summarize, the distribution of dark matter within our galaxy is too sparse to act as a diffusing lens that sig-

nificantly diminishes the amplification of neutrino flux for a supernova lensed by the black hole at the galactic center. Moreover, the setup of a galactic supernova lensed by the galactic center black hole seems the most promising configuration for the observation of gravitational lensing of neutrinos. In most other configurations for which the neutrino flux might be highly amplified, lensing would also be observed in light, whereas supernovae on the other side of the galactic center are only visible in neutrinos.

## V. CONCLUSIONS

Observation of lensed neutrinos emitted by a core-collapse supernova would constitute a graphic demonstration of the gravity's influence on neutrinos. Amplification of the neutrino flux at Earth by many orders of magnitude may occur for near-perfect alignment of supernova, black hole, and Earth, but such events are exceedingly rare. We estimate that a lensing event with magnification by two orders of magnitude has occurred once in the history of the solar system, and the mean time between factor-of-ten events is 250 million years.

A dispersion of the arrival time of supernova neutrinos is a slightly more promising marker for not-quite-perfect alignment. A pulse stretched by more than 20 s has occurred on average once in 29 million years. Neutrino telescope observers should be alert to the interesting possibility of distorted profiles in time for neutrino bursts that emanate from beyond the galactic center.

For all of this, the observation of a spectacular lensing of supernova neutrinos in real time is highly improbable. While "it is a part of probability that many improbable things must happen [75]," we shall have to look elsewhere to make quantitative studies of the gravitational interactions of neutrinos.

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- [1] D. R. Brill and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 465 (1957).
  - [2] A. Einstein, *Science* **84**, 56 (1936).
  - [3] F. W. Dyson, A. S. Eddington, and C. Davidson, *Phil.*

- Trans. A* **220**, 291 (1920).
- [4] R. B. Mann and U. Sarkar, *Phys. Rev. Lett.* **76**, 865 (1996), hep-ph/9505353.
- [5] M. J. Longo, *Phys. Rev. Lett.* **60**, 173 (1988).



- [6] L. M. Krauss and S. Tremaine, Phys. Rev. Lett. **60**, 176 (1988).
- [7] Y. Ascasibar, P. Jean, C. Boehm, and J. Knoedlseder (2005), astro-ph/0507142.
- [8] I. I. Shapiro, Phys. Rev. Lett. **13**, 789 (1964).
- [9] A. R. Cooray, Astron. Astrophys. **348**, 31 (1999), astro-ph/9904246.
- [10] M. Kaplinghat, L. Knox, and Y.-S. Song, Phys. Rev. Lett. **91**, 241301 (2003), astro-ph/0303344.
- [11] J. Lesgourgues, L. Perotto, S. Pastor, and M. Piat, Phys. Rev. **D73**, 045021 (2006), astro-ph/0511735.
- [12] J. Lesgourgues and S. Pastor (2006), astro-ph/0603494.
- [13] R. S. Raghavan (2006), hep-ph/0601079.
- [14] R. V. Pound and G. A. Rebka, Jr., Phys. Rev. Lett. **4**, 337 (1960).
- [15] W. M. Visscher, Phys. Rev. **116**, 1581 (1959).
- [16] W. P. Kells and J. P. Schiffer, Phys. Rev. C **28**, 2162 (1983).
- [17] R. D. Blandford and C. S. Kochanek, in *Dark Matter in the Universe*, edited by J. N. Bahcall, T. Piran, and S. Weinberg (1987), pp. 133–.
- [18] J. A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, 1999).
- [19] W.-M. Yao et al. (Particle Data Group), Journal of Physics G **33**, 1 (2006), URL <http://pdg.lbl.gov>.
- [20] F. Eisenhauer et al., Astrophys. J. **628**, 246 (2005), astro-ph/0502129.
- [21] M. J. Reid, Annu. Rev. Astron. Astrophys. **31**, 345 (1993).
- [22] J. Wambsganss, Living Rev. Rel. **1**, 12 (1998), astro-ph/9812021.
- [23] M. Bartelmann and P. Schneider, Phys. Rept. **340**, 291 (2001), astro-ph/9912508.
- [24] E. Roulet and S. Mollerach, Phys. Rept. **279**, 67 (1997), astro-ph/9603119.
- [25] C. Alcock et al. (MACHO), Astrophys. J. **542**, 281 (2000), astro-ph/0001272.
- [26] L. M. Krauss and T. A. Small, Astrophys. J. **378**, 22 (1991).
- [27] J. D. Barrow and K. Subramanian, Nature **327**, 375 (1987).
- [28] S. Weinberg, *Gravitation and Cosmology: principles and applications of the general theory of relativity* (John Wiley, New York, 1972), §8.7.
- [29] V. S. Berezinsky, T. K. Gaisser, F. Halzen, and T. Stanev, Astropart. Phys. **1**, 281 (1993).
- [30] R. Escribano, J. M. Frère, D. Monderen, and V. Van Elewyck, Phys. Lett. **B512**, 8 (2001), hep-ph/0105211.
- [31] G. L. Fogli, E. Lisi, D. Montanino, and A. Mirizzi, Phys. Rev. **D68**, 033005 (2003), hep-ph/0304056.
- [32] J. F. Beacom and P. Vogel, Phys. Rev. **D60**, 033007 (1999), astro-ph/9811350.
- [33] E. I. Gates, G. Gyuk, and M. S. Turner, Phys. Rev. **D53**, 4138 (1996), astro-ph/9508071.
- [34] K. M. Ferrière, Rev. Mod. Phys. **73**, 1031 (2001), astro-ph/0106359.
- [35] G. A. Tammann, W. Loeffler, and A. Schroeder, Astrophys. J. Suppl. **92**, 487 (1994).
- [36] M. A. Ruderman, Science **184**, 1079 (1974).
- [37] J. R. Ellis and D. N. Schramm, Proc. Natl. Acad. Sci. USA **92**, 235 (1995), hep-ph/9303206.
- [38] J. R. Ellis, B. D. Fields, and D. N. Schramm, Astrophys. J. **470**, 1227 (1996), astro-ph/9605128.
- [39] B. D. Fields and J. R. Ellis, New Astron. **4**, 419 (1999), astro-ph/9811457.
- [40] J. I. Collar, Phys. Rev. Lett. **76**, 999 (1996), astro-ph/9505028.
- [41] L. E. Strigari, J. F. Beacom, T. P. Walker, and P. Zhang, JCAP **0504**, 017 (2005).
- [42] R. M. Bionta, G. Blewitt, C. B. Bratton, D. Casper, A. Cicio, R. Claus, B. Cortez, M. Crouch, S. T. Dye, S. Errede, et al., Phys. Rev. Lett. **58**, 1494 (1987).
- [43] K. Hirata, T. Kajita, M. Koshihara, M. Nakahata, Y. Oyama, N. Sato, A. Suzuki, M. Takita, Y. Totsuka, T. Kifune, et al., Phys. Rev. Lett. **58**, 1490 (1987).
- [44] E. N. Alekseev, L. N. Alekseeva, I. V. Krivosheina, and V. I. Volchenko, in *ESO Workshop on SN 1987 A*, edited by I. J. Danziger (1987), pp. 237–247.
- [45] M. Aglietta, G. Badino, G. Bologna, C. Castagnoli, A. Castellina, V. L. Dadykin, W. Fulgione, P. Galeotti, F. F. Kalchukov, B. Kortchaguin, et al., Europhysics Letters **3**, 1315 (1987).
- [46] M. L. Costantini, A. Ianni, G. Pagliaroli, and F. Vissani (2006), astro-ph/0608399.
- [47] J. M. LoSecco, Phys. Rev. D **38**, 3313 (1988).
- [48] S. Pakvasa, W. A. Simmons, and T. J. Weiler, Phys. Rev. D **39**, 1761 (1989).
- [49] G. G. Raffelt, Nucl. Phys. Proc. Suppl. **110**, 254 (2002), hep-ph/0201099.
- [50] M. P. Munro et al., Astrophys. J. **589**, 225 (2003), astro-ph/0301371.
- [51] Y. Fukuda et al., Nucl. Instrum. Meth. **A501**, 418 (2003), URL <http://neutrino.phys.washington.edu/~superk/>.
- [52] E. Andres et al., Astropart. Phys. **13**, 1 (2000), astro-ph/9906203, URL <http://amanda.uci.edu/>.
- [53] T. Montaruli (IceCube) (2006), astro-ph/0608140, URL <http://icecube.wisc.edu/>.
- [54] J. F. Beacom and P. Vogel, Phys. Rev. **D58**, 053010 (1998), hep-ph/9802424.
- [55] J. Boger et al. (SNO), Nucl. Instrum. Meth. **A449**, 172 (2000), nucl-ex/9910016, URL <http://www.sno.phy.queensu.ca/>.
- [56] C. K. Jung, AIP Conf. Proc. **533**, 29 (2000), hep-ex/0005046, URL <http://ale.physics.sunysb.edu/uno/>.
- [57] A. S. Dighe, M. T. Keil, and G. G. Raffelt, JCAP **0306**, 005 (2003), hep-ph/0303210.
- [58] B. K. Lubasandorzhev, Nucl. Instrum. Meth. **A553**, 282 (2005).
- [59] L. K. Resvanis (NESTOR), J. Phys. Conf. Ser. **39**, 447 (2006), URL <http://www.nestor.org.gr/>.
- [60] V. Flaminio (ANTARES), PoS **HEP2005**, 025 (2006), URL <http://antares.in2p3.fr/>.
- [61] A. Suzuki (KamLAND), Nucl. Phys. Proc. Suppl. **77**, 171 (1999), URL <http://www.awa.tohoku.ac.jp/KamLAND/>.
- [62] G. Bari et al. (LVD), Nucl. Instrum. Meth. **A277**, 11 (1989), URL <http://www.bo.infn.it/lvd/docs.html>.
- [63] J. J. Zach, A. S. J. Murphy, D. Marriott, and R. N. Boyd, Nucl. Instrum. Meth. **A484**, 194 (2002), nucl-ex/0205011.
- [64] C. K. Hargrove, I. Batkin, M. K. Sundaresan, and J. Dubeau, Astropart. Phys. **5**, 183 (1996).
- [65] A. Rubbia, AIP Conf. Proc. **533**, 220 (2000), hep-ex/0001052.
- [66] W. C. Haxton and C. W. Johnson, Nature **333**, 325 (1988).

- [67] G. A. Cowan and W. C. Haxton, *Science* **216**, 51 (1982).
- [68] K. Wolfsberg et al., *AIP Conf. Proc.* **126**, 196 (1984).
- [69] V. T. Nguyen and C. W. Johnson (2005), *astro-ph/0508267*.
- [70] J. J. Binney and N. W. Evans, *Mon. Not. Roy. Astron. Soc.* **327**, L27 (2001), *astro-ph/0108505*.
- [71] B. Moore et al., *Astrophys. J.* **524**, L19 (1999).
- [72] J. F. Navarro, C. S. Frenk, and S. D. M. White, *Astrophys. J.* **490**, 493 (1997), *astro-ph/9611107*.
- [73] R. Escribano, J. M. Frère, D. Monderen, and V. Van Elewyck (1999), *hep-ph/9910510*.
- [74] S. Ando, J. F. Beacom, and H. Yuksel, *Phys. Rev. Lett.* **95**, 171101 (2005), *astro-ph/0503321*.
- [75] G. Eliot, *Daniel Deronda* (1876), Part VI: “Revelations,” Chapter 41, URL <http://www.literature.org/authors/eliot-george/daniel-deronda>